Analysis of a Generalized Retrial System with Coupled Orbits

Evsey Morozov (Institute of Applied Mathematical Research),
Taisia Morozova (Petrozavodsk State University)

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Consider retrial system with **coupled orbits**: retrial (transmission) rate depends on binary state "**busy-idle**" of other orbits.

**Motivation**: In cognitive radio, a wireless node can know state of environment (other sources) and dynamically changes retrial rates to achieve **full spectrum utilization** [2, 3, 4].

**Current trend** to dense networks increases the impact of **wireless interference** [4].
Model description

- Single server queuing system
- Class-$i$ customers follow Poisson input (with parameter $\lambda_i$) and service time $S^{(i)}$ with **general distribution** and parameter
  \[ \gamma_i = \frac{1}{ES^{(i)}} \in (0, \infty), \ i = 1, \ldots, N. \]
- Class-$i$ customer joins the $i$th orbit if server is busy.
- Exponential class-$i$ retrial time with parameter $\mu_i^{(i)}$ depending on current states of other orbits: **busy or empty**.
In more detail, consider \((N - 1)\)-dimensional vector 

\[
J(i) = \{j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_N\},
\]

the component \(j_i\) is omitted.

Each component \(j_k \in \{0, 1\}\): if \(j_k = 1\), then orbit \(k\) is busy, otherwise, if \(j_k = 0\), then orbit \(k\) is empty.

Vector \(J(i)\) describes the states of other orbits \(k \neq i\).
Model description

- We call $J(i)$ a configuration.
- Define set $G(i) = \{ J(i) \} = \{ \text{all possible configurations } J(i) \}$.
- With configuration $J(i)$, orbit $i$ has given constant rate $\mu_{J(i)}$.
- Denote set $M_i = \{ \mu_{J(i)} \}$ - all possible rates when orbit $i$ is busy.
- This model becomes classic multiclass retrial if, for given $i$, $\mu_{J(i)} = \mu_i$ is a constant for all $J(i)$. 
Particular case: 3 orbits

- For 3 orbits: 1, 2, 3, capacity of each $M_i$ and $G(i)$ equals 4.
- Indeed, given orbit $i$, for 2 other orbits $j < k$, the following 4 configurations $J(i)$ are possible:

$$\mathcal{G}(i) = \{ J(i) \} = \{(i_j = 0, i_k = 0), (i_j = 1, i_k = 0), (i_j = 0, i_k = 1), (i_j = 1, i_k = 1) \}.$$  

(1)

- In notation $\mu^i_{(\cdot)}$ the state of configuration $J(i)$ is reflected. For instance, $\mu^1_{01} = \text{rate of configuration } J(1) = (i_2 = 0, i_3 = 1)$, and $\mu^3_{10} = \text{rate of configuration } J(3) = (i_1 = 1, i_2 = 0)$. 

Notation

\[ I(t) = \text{idle time of server in } [0, t]. \]
\[ B(t) = \text{busy time of server in } [0, t], \text{ that is } I(t) + B(t) = t. \]

**Stationary idle and busy probability of server are defined:**

\[
\lim_{t \to \infty} \frac{I(t)}{t} = P_0 = 1 - P_b = 1 - \lim_{t \to \infty} \frac{B(t)}{t}.
\] (2)

load coefficients: \( \rho_i = \lambda_i / \gamma_i, \rho = \sum_{i=1}^{N} \rho_i. \)

**maximal rate of orbit** \( i \) : \( \hat{\mu}_i = \max_{J(i) \in G(i)} \mu_{J(i)}; \)

**minimal rate of orbit** \( i \) : \( \mu^0_i = \min_{J(i) \in G(i)} \mu_{J(i)}. \)

\( B_i(t) = \text{busy time server is occupied by class-} i \text{ customers, in } [0, t]. \)
Theoretical statements

The proofs use balance equations and regenerative method [1, 4].

**Theorem 1.** The stationary probability the server is occupied by class-\(i\) customer is

\[
P_b^{(i)} = \lim_{t \to \infty} \frac{B_i(t)}{t} = \rho_i, \quad i = 1, \ldots, N. \quad (3)
\]

Let \(P_0^{(i)}\) be stationary probability server is idle and orbit \(i\) is busy.

**Theorem 2.** The following bounds hold:

\[
\frac{\lambda_i}{\hat{\mu}_i} \rho \leq P_0^{(i)} \leq \frac{\lambda_i}{\mu_i^0} \rho, \quad i = 1, \ldots, N. \quad (4)
\]
For **classic retrial model** with $\mu_J(i) \equiv \mu_i$ it holds:

**Theorem 3.** Stationary probability that server and orbit $i$ are idle:

$$P_{0,0}^{(i)} = 1 - \rho \left(1 + \frac{\lambda_i}{\mu_i}\right), \quad i = 1, \ldots, N.$$  \hspace{1cm} (5)

**Corollary.** Inequality (4) becomes

$$P_{0}^{(i)} = \frac{\lambda_i}{\mu_i} \rho, \quad i = 1, \ldots, N.$$  \hspace{1cm} (6)
Stability conditions

**Theorem 4.** Necessary stability condition:

\[ P_b = \sum_{i=1}^{N} \rho_i = \rho \leq \min_{1 \leq i \leq N} \left[ \frac{\hat{\mu}_i}{\lambda_i + \hat{\mu}_i} \right] < 1. \]  
(7)

**Theorem 5.** Sufficient stability condition:

\[ \rho \leq \min_{1 \leq i \leq N} \left[ \frac{\mu_i^0}{\lambda_i + \mu_i^0} \right] < 1. \]  
(8)

Introduce the metrics

\[ \Gamma_1 := \min_i \frac{\hat{\mu}_i}{\lambda_i + \hat{\mu}_i} - \rho, \quad \Gamma_2 := \min_i \frac{\mu_i^0}{\lambda + \mu_i^0} - \rho. \]  
(9)

\( \Gamma_i > 0 \) in stability zone, and \( \Gamma_i \leq 0 \), otherwise.
Now we discuss an important special class: symmetric systems. The classic retrial system with constant rate is symmetric, if all corresponding parameters are equal:

$$\lambda_i \equiv \text{constant}, \; \gamma_i \equiv \gamma, \; \mu_i \equiv \mu.$$ 

For system with coupled orbits the notion symmetry is more flexible. Now we illustrate it.
Symmetric model

We illustrate symmetry for model with 3 orbits with rates

\[ M_i = \{ \mu_{00}^i, \mu_{10}^i, \mu_{01}^i, \mu_{11}^i \}, \quad i = 1, 2, 3, \]

corresponding to configurations

\[ \mathcal{G}(i) = \{ J(i) \} = (\{0, 0\}, \{1, 0\}, \{0, 1\}, \{1, 1\}). \]

In symmetric model,

\[ M_1 = M_2 = M_3 \]

but rates within \( M_i \) may be different.
Simulation setting

We consider: 3 orbits; $N(t) = \text{orbit size.}$

($t$-axis counts \textit{discrete events}: arrivals, attempts, departures.)

Exponential $\exp(\gamma_i)$ or Pareto service time distribution:

$$F_i(x) = 1 - \left(\frac{x_0^i}{x}\right)^\alpha, \quad x \geq x_0^i \quad (F_i(x) = 0, \quad x \leq x_0^i),$$

with expectation

$$\frac{1}{\gamma_i} = \mathbb{E}S^{(i)} = \frac{\alpha x_0^i}{\alpha - 1}, \quad \alpha > 1, \quad x_0^i > 0, \quad i = 1, 2, 3. \quad (10)$$
Simulation: symmetric model

For the *symmetric exponential and Pareto models* we choose the following parameters:

\[
\lambda_1 = \lambda_2 = \lambda_3 = 3;
\]

\[
M_1 = \{\mu_{00}^1 = 20, \mu_{10}^1 = 30, \mu_{01}^1 = 15, \mu_{11}^1 = 25\},
\]

\[
M_2 = \{\mu_{00}^2 = 20, \mu_{10}^2 = 30, \mu_{01}^2 = 15, \mu_{11}^2 = 25\},
\]

\[
M_3 = \{\mu_{00}^3 = 20, \mu_{10}^3 = 30, \mu_{01}^3 = 15, \mu_{11}^3 = 25\}. \quad (11)
\]
Simulation: exponential symmetric model

\[ \gamma_1 = \gamma_2 = \gamma_3 = 12, \text{ implying } \Gamma_1 = 0.14, \Gamma_2 = -0.16; \]

**Figure:** 1. Symmetric system, exponential service time. Condition (7) holds, condition (8) is violated: \( \Gamma_1 > 0, \Gamma_2 < 0; \) **but all orbits are stable.**
Simulation: exponential symmetric model

\[ \gamma_1 = \gamma_2 = \gamma_3 = 10, \text{ implying } \Gamma_1 = 0, \Gamma_2 = -0.3; \]

\[ \text{Figure: 3. Symmetric system, exponential service time. Conditions (7) and (8) are violated: } \Gamma_1 < 0, \Gamma_2 < 0; \text{ all orbits are unstable.} \]
For Pareto service time, we select $\alpha = 2$ and shape parameter

$$x_0^i = \frac{1}{30} \quad \text{for orbit } i = 1, 2, 3.$$  \hspace{1cm} (12)

This gives service rates $\gamma_1 = \gamma_2 = \gamma_3 = 15$.

All other parameters remain as in (11).
Simulation: symmetric Pareto model

$\Gamma_1 = 0.3, \Gamma_2 = 0.23$.

**Figure**: 4. Pareto service time. Both conditions (7), (8) hold: $\Gamma_1 > 0, \Gamma_2 > 0$; all orbits are stable.
Simulation: symmetric Pareto model

\[ x_0^i = \frac{1}{12}, \quad \gamma_i = 6, \quad \Gamma_1 = -0.6, \quad \Gamma_2 = -0.67; \]

**Figure:** 5. Pareto service time. Conditions (7) and (8) are violated: \( \Gamma_1 < 0, \Gamma_2 < 0; \) all orbits are unstable.
Simulation: estimation of probability $P_0^{(i)}$

For the next experiments for the exponential model we choose the following parameters:

$$\lambda_1 = \lambda_2 = \lambda_3 = 3;$$
$$\gamma_1 = 10, \gamma_2 = 20, \gamma_3 = 30;$$
$$M_1 = \{\mu_{00}^1 = 20, \mu_{10}^1 = 30, \mu_{01}^1 = 15, \mu_{11}^1 = 20\},$$
$$M_2 = \{\mu_{00}^2 = 10, \mu_{10}^2 = 12, \mu_{01}^2 = 23, \mu_{11}^2 = 30\},$$
$$M_3 = \{\mu_{00}^3 = 25, \mu_{10}^3 = 30, \mu_{01}^3 = 7, \mu_{11}^3 = 20\}.$$  \hspace{1cm} (13)

$$\Gamma_1 = 0.35, \Gamma_2 = 0.28; \text{ Both stability conditions are satisfied.}$$
Simulation: estimation of probability $P^{(1)}_0$

**Figure:** 6. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P^{(1)}_0 = P(\text{busy orbit 1, idle server})$ satisfies bounds (4).
Simulation: estimation of probability $P_0^{(2)}$

Figure: 7. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P_0^{(2)} = P(\text{busy orbit 2, idle server})$ satisfies bounds (4).
Simulation: estimation of probability $P^{(3)}_0$

**Figure:** 8. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P^{(3)}_0 = P(\text{busy orbit 3, idle server})$ satisfies bounds (4).
To verify some theoretical results, we simulate a 3-class retrial system with independent Poisson inputs and the coupled orbits:

- A class-\(i\) customer meeting server busy joins virtual infinite capacity \(i\)-orbit;
- The orbit \(i\) retrial rate depends on configuration of other orbits: \textit{busy or idle}.
- Simulation indicates that the necessary stability condition is indeed stability criterion.


Thanks for your attention!

Taisia Morozova
Petrozavodsk State University
tiamorozova@mail.ru

ResearcherID: J-4140-2018
ORCID: https://orcid.org/0000-0002-3176-2249

ResearchGate: https://www.researchgate.net/profile/Taisia_Morozova