An Algebraic Approach to Scheduling Problems in Project Management

Nikolai Krivulin

Faculty of Mathematics and Mechanics
Saint Petersburg State University

E-mail: nkk<at>math.spbu.ru
URL: http://www.math.spbu.ru/user/krivulin/

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Outline

Motivating Example: Activity Network Model

Start-to-Finish Precedence Relationship

- Consider a project consisting of n activities
- Every activity finishes as soon as some work is performed within some other activities
- For each activity $i = 1, \ldots, n$ we introduce the notation

  x_i, *the initiation time*;

  y_i, *the completion time*;

  a_{ij}, *the time activity j takes to do the work that has to be completed before the completion of activity i*
- The completion time of activity i can be represented as

$$y_i = \max(x_1 + a_{i1}, \ldots, x_n + a_{in})$$

Model Transformation

- Consider the precedence relationship equations

$$y_i = \max(x_1 + a_{i1}, \dots, x_n + a_{in}), \quad i = 1, \dots, n$$

- Substitution of the symbol $\oplus$ for $\max$, and $\otimes$ for $+$ gives

$$y_i = a_{i1} \otimes x_1 \oplus \cdots \oplus a_{in} \otimes x_n, \quad i = 1, \dots, n$$

- With the symbol $\otimes$ omitted, the equations takes the form

$$y_i = a_{i1} x_1 \oplus \cdots \oplus a_{in} x_n, \quad i = 1, \dots, n$$

(note a formal similarity to equations in the conventional algebra

$$y_i = a_{i1} x_1 + \cdots + a_{in} x_n, \quad i = 1, \dots, n)$$

Vector Representation

- The matrix-vector notation

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

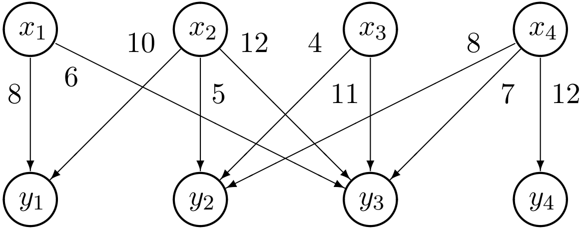
- The precedence relationship equation in the vector form

$$\boldsymbol{y} = A\boldsymbol{x}$$

(matrix-vector multiplication is performed in the usual way with the standard addition and multiplication replaced with $\oplus$ and $\otimes$)

A Network and Its Matrix

- An activity network



- The network precedence relationship matrix ($\mathbb{0} = -\infty$)

$$A = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}$$

Schedule Development Problem

Schedule Development Under Late Finish Date Constraints

- Suppose each activity $i = 1, \ldots, n$ is subject to the time constraint b_i, *the late finish date*
- The vector notation: $\boldsymbol{b} = (b_1, \ldots, b_n)^T$

Problem

- Find the vector $\boldsymbol{x}$ of start dates to meet the condition $\boldsymbol{y} = \boldsymbol{b}$
- The solution satisfies the linear equation of the first kind

$$A\boldsymbol{x} = \boldsymbol{b}$$

in a semiring with the operations $\oplus = \max$ and $\otimes = +$

Idempotent Algebra: Notation and References

Idempotent Semiring $\mathbb{R}_{\max,+}$

- Idempotent semiring (semifield)

$$\mathbb{R}_{\max,+} = \langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$$

- The set: $\mathbb{X} = \mathbb{R} \cup \{-\infty\}$
- The operations: $\oplus = \max$ and $\otimes = +$
- Null and identity elements: $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$
- The inverse: for each $x \in \mathbb{R}$ there exists x^{-1} ($-x$ in conventional algebra)
- The power: for each $x, y \in \mathbb{R}$ one can define x^y (xy in conventional algebra)

Matrix Algebra Over $\mathbb{R}_{\max,+}$

- Addition and multiplication

$$\{A \oplus B\}_{ij} = \{A\}_{ij} \oplus \{B\}_{ij}, \quad \{BC\}_{ij} = \bigoplus_k \{B\}_{ik}\{C\}_{kj}$$

- Identity and null matrices: $I = \operatorname{diag}(\mathbb{1}, \dots, \mathbb{1})$ and $\mathbb{0} = (\mathbb{0})$
- The power: $A^0 = I$, $A^{k+l} = A^k A^l$ for all integer $k, l \geq 0$
- The norm and trace: for any matrix $A = (a_{ij})$

$$\|A\| = \bigoplus_{i,j} a_{ij}, \quad \operatorname{tr} A = \bigoplus_i a_{ii}$$

- The pseudoinvers: for any matrix $A = (a_{ij})$ there exists $A^- = (a_{ij}^-)$ with $a_{ij}^- = a_{ji}^{-1}$, if $a_{ji} \neq \mathbb{0}$, and $a_{ij}^- = \mathbb{0}$, otherwise

Early Publications

- N.N. Vorob'ev (1963), A.A. Korbut (1965), I.V. Romanovskii (1967)

Books

- R.A. Cuninghame-Green (1979), B. Carré (1979)
- U. Zimmermann (1981), F. Baccelli et al (1992)
- V.P. Maslov, V.N. Kolokol'tsov (1994), J.S. Golan (1999)
- B. Heidergott et al (2006), N.K. Krivulin (2009)

Hundreds of Contributing Papers

- V.P. Maslov, G.L. Litvinov, G.B. Shpiz, A.N. Sobolevskii, V.D. Matveenko, S.L. Blyumin
- G.J. Olsder, B. Heidergott, S. Gaubert, B. De Schutter, G. Cohen
- ...

Solution to Example: First Kind Linear Equations

Problem

- Given a $(m \times n)$-matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^m$, find the solution $\boldsymbol{x} \in \mathbb{R}^n$ of the first kind equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

Theorem (Existence and Uniqueness)

1. *The equation has a solution if and only if* $(A(\boldsymbol{b}^{-}A)^{-})^{-}\boldsymbol{b} = \mathbb{1}$
2. *The maximum solution, if any, takes the form* $\boldsymbol{x} = (\boldsymbol{b}^{-}A)^{-}$
3. *If the columns of* A *form a minimal set generating* $\boldsymbol{b}$, *then the solution is unique*

General Solution

- For the matrix A, consider a minimal subset of its columns generating $\boldsymbol{b}$, and denote the set of the column indices by J
- Let $\mathcal{J}$ be the set of all the subsets J
- Let G_J be the diagonal matrix that has its diagonal entry in row i set to $\mathbb{0}$, if $i \in J$, and to $\mathbb{1}$, otherwise

Theorem

The general solution of the first kind equation is the family

$$\boldsymbol{x}_J = (\boldsymbol{b}^{-} A \oplus \boldsymbol{v}^T G_J)^{-}, \quad \boldsymbol{v} \in \mathbb{R}^n, \quad J \in \mathcal{J}$$

Corollary

The solution of the inequality $A\boldsymbol{x} \leq \boldsymbol{b}$ *is given by* $\boldsymbol{x} \leq (\boldsymbol{b}^{-} A)^{-}$

Example 2: Activity Network Model

Start-to-Start Precedence Relationship

- A project involves n activities
- Every activity starts not earlier than some work is performed within some other activities
- For each activity $i = 1, \ldots, n$ we introduce the notation

  x_i , *the initiation time*;

  y_i , *the completion time*;

  a_{ij} , *the time activity j takes to do the work that has to be completed before the start of activity i*
- The initiation time of activity i satisfies the condition

$$x_i \geq \max(x_1 + a_{i1}, \ldots, x_n + a_{in})$$

Model Representation

- In terms of $\mathbb{R}_{\max,+}$, the precedence relationships take the form

$$x_i \geq a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n, \quad i = 1, \ldots, n$$

- With the matrix-vector notation, we arrive at the inequality

$$A\boldsymbol{x} \leq \boldsymbol{x}$$

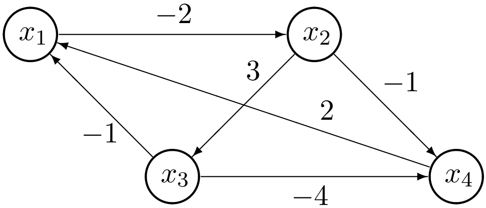
Problem

- Find the vector $\boldsymbol{x}$ that satisfies the precedence constraints
- Of particular interest is the solution of the homogeneous linear equation of the second kind

$$A\boldsymbol{x} = \boldsymbol{x}$$

A Network and Its Matrix

- An activity network



- The network precedence relationship matrix ($\mathbb{0} = -\infty$)

$$A = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}$$

Schedule Development Problem

Schedule Development Under Early Start Date Constraints

- Suppose each activity $i = 1, \ldots, n$ is subject to the time constraint b_i, *the early start date*
- The vector notation: $\boldsymbol{b} = (b_1, \ldots, b_n)^T$

Problem

- Find a vector $\boldsymbol{x}$ so as to meet the conditions

$$A\boldsymbol{x} = \boldsymbol{x}, \quad \boldsymbol{x} \geq \boldsymbol{b}$$

- The solution satisfies the nonhomogeneous linear equation of the second kind

$$A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

Linear Equations of the Second Kind

Problem: Solution for Homogeneous Bellman Equation

- Given a $(n \times n)$-matrix A, find a solution $\boldsymbol{x} \in \mathbb{R}^n$ of the equation

$$A\boldsymbol{x} = \boldsymbol{x}$$

Problem: Solution for Nonhomogeneous Bellman Equation

- Given a $(n \times n)$-matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^n$, find a solution $\boldsymbol{x} \in \mathbb{R}^n$ of the equation

$$A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

Solution

- For each $(n \times n)$ -matrix A, we introduce the matrices

$$A^{+} = I \oplus A \oplus \cdots \oplus A^{n-1}, \qquad A^{\times} = AA^{+} = A \oplus \cdots \oplus A^{n},$$

and the symbol

$$\operatorname{Tr} A = \bigoplus_{m=1}^{n} \operatorname{tr} A^{m}$$

- Provided that $\operatorname{Tr} A = \mathbb{1}$, we define the matrix A^{*} with the columns

$$\boldsymbol{a}_i^{*} = \begin{cases} \boldsymbol{a}_i^{+}, & \text{if } a_{ii}^{\times} = \mathbb{1}, \\ \mathbb{0}, & \text{otherwise}, \end{cases}$$

where $\boldsymbol{a}_i^{+}$ is column i of A^{+}, and $a_{ii}^{\times}$ is entry (i, i) of $A^{\times}$

Lemma

Let $\boldsymbol{x}$ be the general solution of the homogeneous equation with an irreducible matrix. Then it holds

1) *if* $\mathrm{Tr}\,A = \mathbb{1}$, *then* $\boldsymbol{x} = A^{*}\boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}^{n}$;
2) *if* $\mathrm{Tr}\,A \neq \mathbb{1}$, *then there exists only the solution* $\boldsymbol{x} = \mathbb{0}$

Theorem

Let $\boldsymbol{x}$ be the general solution of the nonhomogeneous equation with an irreducible matrix. Then it holds

1) *if* $\mathrm{Tr}\,A < \mathbb{1}$, *then there exists the unique solution* $\boldsymbol{x} = A^{+}\boldsymbol{b}$;
2) *if* $\mathrm{Tr}\,A = \mathbb{1}$, *then* $\boldsymbol{x} = A^{+}\boldsymbol{b} \oplus A^{*}\boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}^{n}$;
3) *if* $\mathrm{Tr}\,A > \mathbb{1}$, *then with the condition* $\boldsymbol{b} = \mathbb{0}$, *there exists only the solution* $\boldsymbol{x} = \mathbb{0}$, *whereas with* $\boldsymbol{b} \neq \mathbb{0}$ *there is no solution*

Example 3: Schedule Development Problem

Schedule Development Under Mixed Time Constraints

- Consider a project with late finish date constraints in the form

$$A_1 \boldsymbol{x} \leq \boldsymbol{b}$$

- Suppose the project also has early start date constraints imposed

$$A_2 \boldsymbol{x} = \boldsymbol{x}$$

Problem

- Find the vector $\boldsymbol{x}$ to meet the mixed set of precedence constraints

Solution

- Suppose the equation $A_2\boldsymbol{x} = \boldsymbol{x}$ has the solution

$$\boldsymbol{x} = A_2^{*}\boldsymbol{v}$$

- Substitution of the solution into the inequality $A_1\boldsymbol{x} \leq \boldsymbol{b}$ gives

$$A_1 A_2^{*}\boldsymbol{v} \leq \boldsymbol{b}$$

- The maximum solution of the last inequality takes the form

$$\boldsymbol{v} = (\boldsymbol{b}^{-} A_1 A_2^{*})^{-}$$

- Therefore, the vector x of activity initiation dates is written as

$$\boldsymbol{x} = A_2^{*}(\boldsymbol{b}^{-} A_1 A_2^{*})^{-}$$

Conclusions

- A new approach to schedule development is proposed based on idempotent algebra
- The approach offers a convenient algebraic technique to describe and analyze different logical relationships in schedules
- The approach reduces scheduling problems to solution of linear equations in an idempotent semiring
- The solutions to the equations are given in compact vector form
- The approach and related techniques provide the basis for new efficient software solutions for schedule development

Acknowledgments

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