Regenerative simulation of the loss probability in the finite buffer queue with Brownian input

Ruslana S. Goricheva†, Oleg V. Lukashenko†, Evsey V. Morozov†, Michele Pagano‡

†Karelian Research Center RAS, Petrozavodsk State University

‡University of Pisa

11, Pushkinskaya Street, Petrozavodsk, Karelia, Russia, 185640
Via Caruso,16, Pisa, Italy, I-56126

Abstract
We discuss the application of the regenerative simulation to estimate the loss probability in a queueing system with finite buffer which is fed by a Brownian input (Bi). Some numerical examples are also included. This work is supported by Russian Foundation for Basic research, project No 10-07-00017 and done in the framework of the Strategy development Program for 2012-2016 "PetrSU complex in the research-educational space of European North: strategy of the innovation development.

1 Introduction

In this work we are interested in systems with small or moderate size buffers, because it is motivated by real network applications, which have stringent requirements to queueing delay. So the loss rate prediction can be useful to provide suitable level of Quality of Service.

To motivate our interest to systems fed by Bi, we note that appropriately scaled superposition of large number of identically distributed (i.i.d.) on-off sources with finite variances converges weakly to Brownian motion (Bm) provided first, number of sources $M \to \infty$, and then scaling factor $T \to \infty$ (see [5, 8] for more details).

This result gives formal motivation for the following definition of Bi, which is used below:

$$A(t) = mt + \sqrt{am}B(t), \tag{1.1}$$

where m is the mean input rate, and Bm $\{B(t),\, t \geq 0,\}$ describes random fluctuations of the input around its linearly increasing mean, a - some constant, see [6].

2 Queue with Brownian input

It is known that the workload $Q(t)$ can be calculated by the following Lindley-type recursion for the finite buffer system [6]:

$$Q(t) = \min((Q(t-1) - C + m + \sqrt{am}(B(t) - B(t-1)))^+, b), \ t = 1, 2, \cdots \quad (2.1)$$

where $(x)^+ = \max(0, x)$.

A typical sample path of the workload process (2.1) is presented in Figure 1 (where $C = 1$, $m = 0.7$, $b = 3$).

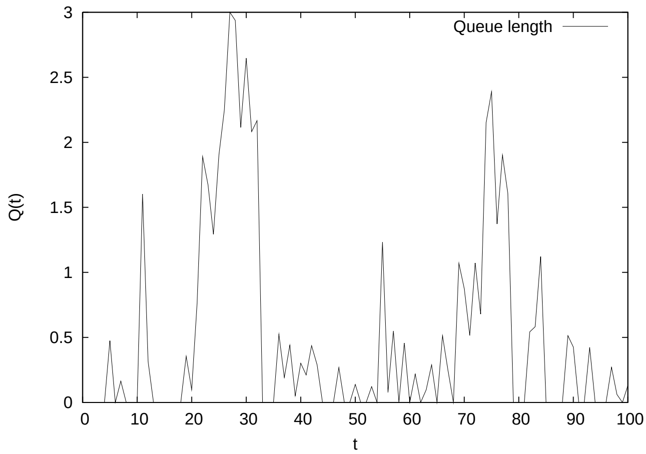

Figure 1: Finite buffer system with Bi sample path

3 Regenerative method

In this section, we describe in brief the method of regenerative simulation and the weakest known condition under which the regenerative method can be applied for the confidence estimation. A process $X = \{X_t, t \in T\}$, where $T = [0, \infty)$ (or $T = \{0, 1, \dots\}$) is called *regenerative process* if there exists an infinite sequence of instants $0 = \beta_0 < \beta_1 < \beta_2 < \cdots$ (regeneration points) such that the segments $G_n = (X_t, \ \beta_{n-1} \le t < \beta_n)$ (regeneration cycles) are i.i.d. The cycle periods $\beta_{n+1} - \beta_n$, $n \ge 0$, are also i.i.d. and we denote by β generic regeneration period.

For definiteness, we consider a discrete-time positive recurrent process X (that is $\mathsf{E}\beta < \infty$) and assume that regeneration cycle length β is aperiodic ($\mathsf{P}(\beta = 1) > 0$). Then the weak limit $X_n \Rightarrow X$ as $n \to \infty$ exists such that $\mathsf{P}(X < \infty) = 1$. Moreover, if f is a measurable function, then the following statement holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(X_i) = \frac{\mathsf{E}[\sum_{i=0}^{\beta-1} f(X_i)]}{\mathsf{E}\beta} \equiv r.$$

It is also assumed that $\mathsf{E}[\sum_{i=0}^{\beta-1} |f(X_i)|] < \infty$. (Note that an evident analog for continuous-time process also exists. More details can be found in [1].)

To estimate the unknown parameter r (steady-state performance measure), we group the data belonging to the same regenerative cycle to obtain the i.i.d. enlarged variables

$$Y_k := \sum_{n=\beta_{k-1}}^{\beta_k - 1} f(X_n), \qquad k \geq 1.$$

We now define the main sample-mean ratio-type estimator as follows:

$$r_n \equiv \frac{\overline{Y}_n}{\overline{\alpha}_n},$$

where $\overline{\alpha}_n$ is the sample mean cycle period,

$$\overline{Y}_n = \frac{1}{n} \sum_{i}^{n} Y_i,$$

and n is the number of completed regeneration cycles obtained during simulation. Let us also denote the variance of the enlarged variable as $\sigma^2 = \mathsf{E}(Y - r\beta)^2$.

If the (minimal sufficient) condition

$$0 < \mathsf{E}(Y - r\beta)^2 < \infty$$

holds, then the following *Regenerative Central Limit Theorem* can be applied [2]:

$$n^{1/2}\overline{\alpha}_n[r_n - r] \Rightarrow \sigma N(0,1), \qquad n \to \infty.$$

This leads to the following $100(1-\gamma)\%$ asymptotic confidence interval for the unknown (steady-state) performance measure r:

$$\left[r_n - \frac{z_\gamma s(n)}{\overline{\alpha}_n \sqrt{n}}, \;\; r_n + \frac{z_\gamma s(n)}{\overline{\alpha}_n \sqrt{n}} \right], \tag{3.1}$$

where

$$\mathsf{P}\left(N(0,1) > z_\gamma\right) \; = \; \frac{1-\gamma}{2}$$

and $s^2(n)$ is the empirical variance, which converges with probability 1 to the variance

$$s^2(n) \to \sigma^2,$$

when the number of observed regeneration cycles $n \to \infty$.

4 Regenerative structure and simulation results

Using regenerative approach, we present the way of loss rate estimation, which can be applied in system $Bi/D/1/n$. First we construct regeneration points for the content process. (More details can be found in [3].) Let $\beta_0 = 0$ and

$$\beta_{k+1} = \min\{t > \beta_k : Q(t-1) = 0,\, Q(t) > 0,\ k \geq 1\}, \tag{4.1}$$

where $Q(t)$ is the queue content at the end of slot t. It is important to stress that in continuous-time setting construction of regenerations meets a difficulty caused by structure of the Brownian input paths [4].

Now we denote by $L_b(t)$ the total load lost in time interval $[0,\, t]$. Denote by $\mathsf{E}L$ the mean load lost per regenerative cycle and let $\mathsf{E}A$ be the mean load arrived per regenerative cycle. Applying regenerative method, we obtain the following representation for the steady-state loss probability

$$\lim_{t\to\infty} \frac{L_b(t)}{A(t)} = \frac{\mathsf{E}L}{\mathsf{E}A} := P_\ell.$$

To apply confidence estimation based on regenerative central limit theorem to estimate probability $P_\ell := r$, we treat processes $\{L_b(t),\, t \geq 0\}$ and $\{A(t),\, t \geq 0\}$ as *cumulative processes* with embedded regenerations defined by recursion (4.1), see [7]. Then we use regenerative simulation to estimate limit ratios

$$r_1 := \lim_{t\to\infty} \frac{L_b(t)}{t} = \frac{\mathsf{E}L}{\mathsf{E}\beta},\ r_2 := \lim_{t\to\infty} \frac{A(t)}{t} = \frac{\mathsf{E}A}{\mathsf{E}\beta},$$

separately, and then use equality $r = r_1/r_2$. Denote by $I_i = [a_i,\, b_i]$ confidence interval (with a confidence level $1-\alpha$) for r_i, $i = 1, 2$. Then it is easy to see that unknown parameter r is covered by the confidence interval $I = [a_1/b_2,\, b_1/a_2]$ (provided $a_2 > 0$) with probability

$$\begin{aligned}
\mathsf{P}(r \in I) &\geq \mathsf{P}(r_i \in I_i,\, i = 1, 2) \\
&\geq \frac{1}{2}\Big(\mathsf{P}(r_1 \in I_1) + \mathsf{P}(r_2 \in I_2) - \mathsf{P}(r_1 \notin I_1) - \mathsf{P}(r_2 \notin I_2)\Big) \\
&= 1 - 2\alpha.
\end{aligned}$$

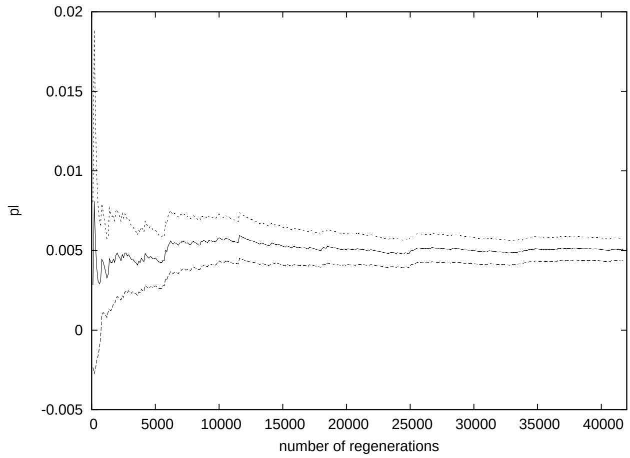

Figure 2: 90% Confidence interval for P_ℓ in Bi/D/1/4

In our experiments we have used $\alpha = 0.05$, so resulting confidence interval has level 90%.

Figure 2 shows 90% confidence interval for the loss probability in the system $Bi/D/1/b$ with Brownian input (with rate $m = 0.7$, service rate $C = 1$ and buffer size $b = 4$) as a function of the simulation length (in terms of regeneration cycles).

Figure 3 shows 90% confidence interval for the loss probability as a function of the buffer size. The following parameters are used: $C = 1$; $m = 0.7$; $N = 10^6$ (where N denotes the number of simulated time slots).

Finally, figure 4 shows 90% confidence interval for the loss probability as a function of the service rate. The following parameters are used: $b = 4$; $m = 0.7$; $N = 10^6$.

To explain an increasing of the confidence length on Figures (3), (4), we recall that use log scale. Moreover, shift below of the center of the intervals caused by decreasing of the loss probability as the buffer size increases.

We note that using regenerations leads to a reliable estimation due to the i.i.d. property of the regeneration cycles.

5 Conclusion

The results of our study can be summarised as follows:

A key observation is that Brownian process has i.i.d. increments and it allows us to construct confidence interval for the loss probability based on simulation of the i.i.d. regeneration cycles. Using such a simulation we

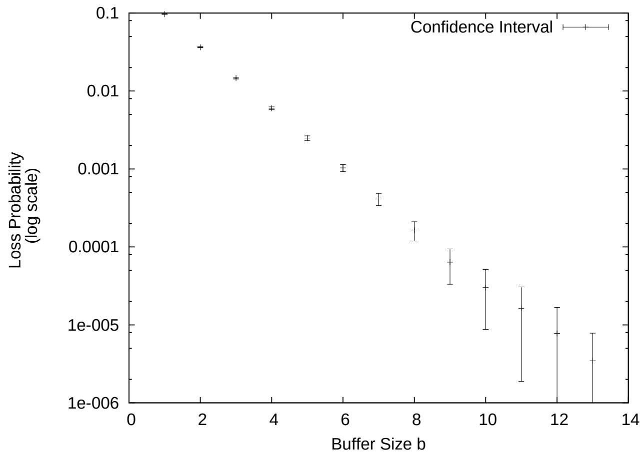

Figure 3: Dependence of log estimate of P_ℓ on buffer size b

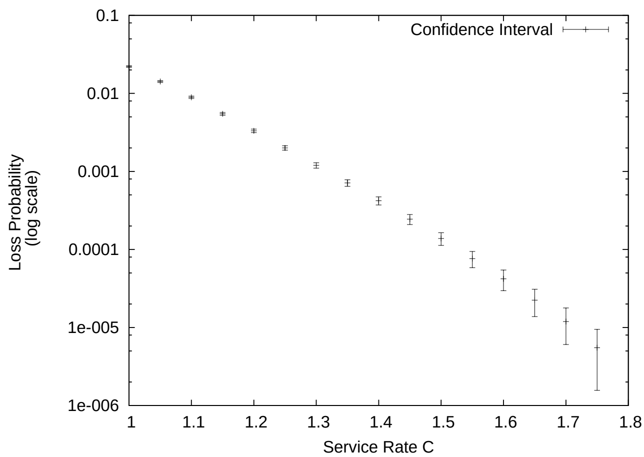

Figure 4: Dependence of log estimate of P_ℓ on service rate C

present a few numerical examples.

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