Self-Similarity and Long-Range Dependence in Network Traffic Modeling

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Abstract

We give a survey of main notions and basic features of self-similar and long-range dependent processes arising in the modeling of modern broadband communication networks. We concentrate on the mathematical properties of the models in such a way to facilitate the practical utilization of these effects in the analysis of network traffic.

1 Self-Similarity

First we give the following definition, [4, 8].

Definition 1. A process $B_H = (B_H(t),\, t \in (-\infty, \infty) = \mathbb{R})$ is called the *normalized fractional Brownian motion (FBM)* with *self-similarity (Hurst) parameter* $H \in (1/2, 1)$ if the following assumptions are satisfied:

(i) B_H has stationary increments;

(ii) $B_H(0) = 0$ and $\mathbf{E}\, B_H(t) = 0$ for all t;

(iii) $\mathbf{E}\, B_H(t)^2 = |t|^{2H}$ for all t;

(iv) B_H has continuous paths;

(v) B_H is a Gaussian process.

In the limiting case $H = 1/2$ process B_H would be *standard Brownian motion (BM)*, and with $H = 1$, B_H would be a deterministic process with linear paths, [9].

By properties (i), (v), finite-dimensional distributions of B_H are fully determined by the (zero) mean and variance function.

The process B_H is an important case of general *self-similar process* $Z_H = (Z_H(t)\,,\ t \in \mathbb{R})$ with self-similarity parameter $H \in (0, 1)$ which is defined as follows: finite-dimensional distributions of the scaled process

$$Z_H^{(T)} = \left(T^{-H} Z_H(Tt)\,,\ t \in \mathbb{R}\right) \tag{1.1}$$

do not depend on *time scale parameter* $T > 0$.

A process Z_H is called *second order self-similar* if the covariance function of the process $Z_H^{(T)}$ does not depend on T. (We note that for a BM this definition is equivalent to self-similarity.)

It is easy to calculate the covariance function of FBM, which is positive and has the form

$$\begin{aligned}\operatorname{cov}(B_H(s), B_H(t)) &= \mathbf{E}\left[B_H(s)B_H(t)\right]\\ &= 1/2\, \mathbf{E}\left[B_H(t)^2 + B_H(s)^2 - (B_H(t) - B_H(s))^2\right]\\ &= 1/2\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\,, \quad t,\, s \in \mathbb{R}\,.\end{aligned} \tag{1.2}$$

We note that if B_H is a BM (that is $H = 1/2$) then from (1.2) or by independence of increments we obtain

$$\operatorname{cov}\left(B_{1/2}(s), B_{1/2}(t)\right) = \min(t, s)\ .$$

Expression (1.2) implies that for any $t_1 < t_2 \le t_3 < t_4$

$$\begin{aligned}&\operatorname{cov}(B_H(t_2) - B_H(t_1),\, B_H(t_4) - B_H(t_3))\\ &\quad = 1/2\left[(t_4 - t_1)^{2H} - (t_3 - t_1)^{2H} + (t_3 - t_2)^{2H} - (t_4 - t_2)^{2H}\right]\,.\end{aligned} \tag{1.3}$$

The (stationary) discrete time sequence of increments of FBM B_H, $(B^*_H(n) = B_H(n+1) - B_H(n),\, n = 0, 1 \dots)$, is called *fractional Gaussian noise* (FGN). It follows from (1.3) that the covariance function of FGN (which coincides with correlation) has the form

$$r(n) = \operatorname{cov}(B^*_H(0), B^*_H(n)) \\ = 1/2 \left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right], \quad n \geq 1 . \quad (1.4)$$

To obtain asymptotic behaviour of $r(n)$ as $n \to \infty$, we apply Taylor's expansion to power functions in (1.4). It gives

$$(n \pm 1)^{2H} = n^{2H} \pm 2Hn^{2H-1} + H(2H-1)n^{2H-2} \\ \pm \frac{2H(2H-1)(2H-2)n^{2H-3}}{3!} + o(n^{2H-3}), \quad (1.5)$$

and then (1.4), (1.5) imply

$$r(n) = H(2H-1)n^{-2(1-H)} + o(n^{-(3-2H)}) \text{ as } n \to \infty , \quad (1.6)$$

or

$$r(n) \sim \text{const} \cdot n^{-2(1-H)} \text{ as } n \to \infty , \quad (1.7)$$

where $\sim$ means that ratio converges to 1. This expression is a basis to introduce more general *asymptotically self-similar process* $Z_H = (Z_H(n),\, n = 0, 1, \dots)$, where covariance function $r(n)$ has the asymptotic form

$$r(n) \sim n^{-\beta}L(n) \text{ as } n \to \infty , \quad (1.8)$$

where parameter $0 < \beta < 1$ and L is *slowly varying function* at infinity (that is $\lim_{n\to\infty} L(nx)/L(n) = 1$ for all $x > 0$). It follows from (1.7) that

$$H = 1 - \frac{\beta}{2} . \quad (1.9)$$

The above stated results can be used to model new surprising features which have been revealed recently by careful statistical analysis of traffic data in modern broadband communication networks (e.g. Ethernet LAN, [14]).

Let $Z = (Z_n, n = 0, 1, \dots)$ be *wide sense stationary (covariance stationary)* sequence with $\mathbf{E}\, Z_0 = 0$, $\mathbf{Var}(Z_0) = \sigma^2$ and *autocorrelation function*

$$r(n) = \frac{\operatorname{cov}(Z_0, Z_n)}{\sigma^2}, \quad n \geq 0 ,$$

which satisfied assumption (1.8). In the network traffic context, we may treat Z_n as the number of packets (bytes) arriving in network traffic per nth time unit. Based on the sequence Z, let us define the aggregated (also covariance stationary) sequence $Z^{(m)} = (Z_n^{(m)}, n \geq 0)$, constructed by partition of original process into non-overlapping blocks of a fixed size m, where

$$Z_n^{(m)} = \frac{Z_{nm} + \cdots + Z_{nm+m-1}}{m^H}, \ \ n \geq 0, \tag{1.10}$$

where parameter $H = 1 - \beta/2 \in (1/2, 1)$, and let $r^{(m)}$ be the autocorrelation function of the process $Z^{(m)}$. Denote $\tilde{Z}_n^{(m)} = Z_{nm} + \cdots + Z_{nm+m-1}$, $n \geq 0$.

As in continuous time, we say that the original process Z is (*exactly) self-similar* with self-similarity parameter $H = 1 - \beta/2$ if finite-dimensional distributions of the aggregated process $Z^{(m)}$ do not depend on block size m. (In particular, process (1.10) is equivalent to Z.) *Second order self-similarity* means that the covariance function of process (1.10) does not depend on m. It is easy to see that for self-similar process Z,

$$\mathbf{Var}\left(Z_0^{(m)}\right) = \sigma^2 m^{-\beta}, \quad m \geq 1 . \tag{1.11}$$

Moreover, autocorrelation function $r^{(m)}$ of the self-similar process $Z^{(m)}$ has form (1.4). To prove this we first note that

$$r(1) = \frac{1}{2} \frac{\mathbf{E}\left[(Z_0 + Z_1)^2 - Z_0^2 - Z_1^2\right]}{\sigma^2} = 2^{2H-1} - 1 ,$$

and then apply induction to obtain representation (1.4). Since by (1.10),

$$\operatorname{cov}\left(\tilde{Z}_0^{(m)}, \tilde{Z}_n^{(m)}\right) = m^{2H} \operatorname{cov}(Z_0, Z_n) ,$$

then (1.11) implies that $r(n) = r^{(m)}(n)$ for all m and n.

We note that a stationary process Z is called *asymptotically second-order self-similar* if its autocorrelation function

$$r^{(m)}(n) \to \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right] \quad \text{as} \quad m \to \infty \ . \tag{1.12}$$

The most unusual feature of self-similar process Z is that the aggregated process $Z^{(m)}$ has a nondegenerate correlation structure as aggregation size $m \to \infty$, in strong contrast to the conventional models of time series, where for each n,

$$r^{(m)}(n) \to 0 \quad \text{as} \quad m \to \infty \ , \tag{1.13}$$

(see [14]). To show this, let us consider an important particular case when sequence Z is k*-dependent* for some fixed $k \geq 0$ (that is $(Z_{n+i},\, i \geq k+1)$ and $(Z_i,\, i \leq n)$ are independent for each $n \geq 0$). We note that independent variables are 0-dependent. It is known, [3], that in this case

$$\frac{\mathsf{Var}\left(\tilde{Z}_0^{(m)}\right)}{m} \to \mathsf{Var}\,(Z_0) \quad \text{as} \quad m \to \infty \ . \tag{1.14}$$

But it is obvious that $\text{cov}\left(Z_0^{(m)}, Z_n^{(m)}\right) = 0$ for $n \geq 2$ and $m \geq k$, that together with (1.14) leads to (1.13).

2 Long-Range Dependence

In this section, we consider some properties of long-range dependent processes and their ties with the self-similarity.

Let $Z = (Z_n,\, n = 1, 2 \ldots)$ be a (covariance) stationary discrete time process with autocorrelation function r. If

$$\sum_{n \geq 1} r(n) = \infty \ ,$$

then the process Z is said to be *long-range dependent* (LRD), otherwise, Z is *short-range dependent.*

First we note that under asymptotic assumption (1.8) the process Z exhibits long-range dependence. Really, due to (1.8),

$$\sum_{n\geq 1} n^{-\beta} L(n) = \infty \ , \tag{2.1}$$

since $\beta < 1$ and summability/nonsummability of the series does not depend on slowly varying function L, [12]. Moreover, the short-range dependent process is characterized by an *exponential decay* of autocorrelation function, that is

$$r(n) \sim \rho^n \quad \text{as} \quad n \to \infty \tag{2.2}$$

for some $\rho \in (0,1)$, [14]. This shows, that relation (1.8) may be used as definition of the LRD process, which is thus determined by a *hyperbolic decay* of autocorrelation function. One can note that divergence in (2.1) guarantees a nondegenerate correlation structure of the aggregated process $Z^{(m)}$ in (1.9) for all m (see (1.12) and also [14]).

The long-range dependence of (wide sense) continuous time stationary process $Z = (Z_t, \, t \in \mathbb{R})$ can also be defined via its autocorrelation function

$$r(t) = \frac{\operatorname{cov}(Z(0), Z(t))}{\mathbf{Var}(Z(0))} \ , \quad t \in \mathbb{R} \ ,$$

by the condition

$$\int_0^\infty r(t) dt = \infty \ . \tag{2.3}$$

(Otherwise, the process Z is short-range dependent.)

Now we discuss some applications of LRD processes which are quite important to model properties of the traffic in the modern communication networks, [7, 9, 10, 14].

Consider independent nonnegative random variables $\tau_1, \tau_2, \ldots$ and let $\{\tau_n, \, n \geq 2\}$ be i.i.d. with distribution function (d.f.) F and $\mathbf{E}\,\tau = \mu < \infty$ (τ is the generic variable of the sequence). Suppose also that

$$F_1(x) = \mathbf{P}(\tau_1 \leq x) = \frac{1}{\mu} \int_0^x (1 - F(u)) du \ , \quad x \geq 0 \ , \tag{2.4}$$

that is the renewal process, generated by the sequence $\{\tau_n,\, n \geq 1\}$ is *stationary*, [1]. Denote $S_0 = 0$, $S_n = \tau_1 + \ldots + \tau_n$, $n \geq 1$, and let $Z = (Z_n,\, n \geq 1)$ be a sequence of i.i.d random variables independent of $\{\tau_n\}$ with $\mathbf{Var}(Z_1) = \sigma^2 < \infty$. Let

$$\Lambda(t) = \begin{cases} Z_n\ , & \text{if } \ t \in [S_{n-1}, S_n)\ ,\ \ n \geq 1\ , \\ 0\ , & \text{otherwise.} \end{cases} \tag{2.5}$$

The process $\Lambda = (\Lambda(t),\, t \geq 0)$ is called *renewal rate process*, [7]. We recall that the underlying renewal process

$$N(t) = \min(n : S_n > t)\ , \quad t \geq 0$$

is stationary, so the process Λ is stationary too. Based on representation

$$\Lambda(t) = \sum_{n \geq 1} Z_n \, \mathbf{1}_{\{t \in [S_{n-1}, S_n)\}}\ ,$$

($\mathbf{1}$ means indicator), we obtain that

$$\mathbf{E}\,\Lambda(t) = \mathbf{E}\,Z_1\ , \quad \mathbf{Var}(\Lambda(t)) = \sigma^2 \ \text{ for all } \ t\ . \tag{2.6}$$

Moreover, applying independence the process Z and indicators $\mathbf{1}$, and also independence $\Lambda(t)$ and $\Lambda(0) = Z_1$ on the event $\{\tau_1\}$ we obtain that (see [7])

$$\begin{aligned} \text{cov}\,(\Lambda(0), \Lambda(t)) &= \mathbf{Var}(Z_1)\,\mathbf{P}(\tau_1 > t) \\ &= \frac{\sigma^2}{\mu} \int_t^\infty (1 - F(u))du\ , \quad t \geq 0\ . \end{aligned} \tag{2.7}$$

Now it follows from (2.4), (2.6), (2.7) that

$$r(t) = \frac{\text{cov}\,(\Lambda(0), \Lambda(t))}{\sigma^2} = \frac{1}{\mu} \int_t^\infty (1 - F(u))du\ , \quad t \geq 0\ . \tag{2.8}$$

In particular, if $\mathbf{E}\,\tau^2 = \infty$ then

$$\int_0^\infty r(t)dt = \frac{1}{\mu} \int_0^\infty u(1 - F(u))du = \infty\ , \tag{2.9}$$

and renewal rate process Λ is LRD.

Let us consider another example (adopted from [7]), where the auto-correlation function is easy to investigate; let $\{\nu(t),\, t \geq 0\}$ be the process counting the number of currently busy servers in infinite server queueing model $M/G/\infty$. Suppose that input is Poisson with intensity λ and i.i.d. service times $\{S_n,\, n \geq 1\}$ have finite mean. Since this model has equilibrium distribution we suppose that the process ν is initially stationary. Then it is well-known that

$$\mathbf{E}\,\nu(0) = \lambda\,\mathbf{E}\,S\ , \tag{2.10}$$

where S is the generic variable for service time. To compute covariance function of the process ν we denote (for a fixed moment t)

$Z_1(t) = \#$ calls in system both at 0 and t;

$Z_2(t) = \#$ calls in system at 0 but not t;

$Z_3(t) = \#$ calls in system at t but not 0.

It is known (for instance, [1]) that this model has Poisson departure process. Thus all Z_i are independent, Poisson distributed variables and

$$\nu(0) = Z_1(t) + Z_2(t)\ ,\quad \nu(t) = Z_1(t) + Z_3(t)\ .$$

This implies

$$\operatorname{cov}(\nu(0), \nu(t)) = \mathbf{Var}(Z_1(t)) = \mathbf{E}\,Z_1(t)\ . \tag{2.11}$$

Let $\mathcal{B}$ be the set of numbers of initially occupied servers, so the capacity of $\mathcal{B}$ equals $\nu(0)$. Let $S(i)$ be the (stationary) residual service time of a server $i \in \mathcal{B}$. By stationarity,

$$\mathbf{P}\,(S(i) > t) = \frac{1}{\mathbf{E}\,S}\int_t^\infty (1 - F(u))du\ ,\quad i \in \mathcal{B}\ , \tag{2.12}$$

and moreover, $\mathbf{E}\,Z_1(t) = \mathbf{E}\left(\sum_{i\in\mathcal{B}} \mathbf{1}_{\{S(i)>t\}}\right)$, where indicators are independent of the set $\mathcal{B}$. Hence (2.10), (2.11) and Wald's identity imply

$$\operatorname{cov}(\nu(0), \nu(t)) = \mathbf{E}\,Z_1(t) = \lambda\int_t^\infty (1 - F(u))du\ , \tag{2.13}$$

(also see [9]). Thus, if $\mathbf{E}\,S^2 = \infty$ then the process ν is LRD.

It would be interesting to apply this approach to queueing regenerative processes where an underlying renewal process is generated by the regeneration points with finite mean regeneration cycle length and infinite variance. The main difficulty here is that usually (unlike the model $M/G/\infty$) values of a regenerative queueing process strongly depend on the current regeneration cycle length and as a result a characteristic under investigation also has infinite variance. In other words, in this case a queueing process treated as the process Z (see (2.5)) would be dependent on the underlying renewal process $\{\tau_n\}$. Nevertheless, it is possible to consider binary-type renewal rate processes, where sequence Z may be treated, for instance, as a "fraction of ON- or OFF- time" during regeneration cycle, and so on (see ON/OFF model in next section).

3 Fractional Brownian Traffic and ON/OFF sources with Heavy Tails

In this section, we consider the so-called ON/OFF model with m sources of *packet trains* and show their role in the understanding of self-similarity and long-range dependence as important features of the traffic in modern communication networks, [13, 11]. First we give the following definition [8, 9].

Definition 2. The process

$$A(t) = mt + \sqrt{am}B_H(t)\ , \quad t \geq 0\ , \tag{3.1}$$

where B_H is a normalized FBM, is called *fractional Brownian traffic.*

The process A has three parameters: Hurst parameter $H \in [1/2, 1)$ (if $H = 1/2$, we call A *Brownian traffic*), mean input rate $m > 0$ and a variance coefficient $a > 0$. To motivate choice (3.1) we may first refer to diffusion approximation of Poisson input N with intensity rate λ, where $N(t)$ is the number of arrivals in interval $[0, t)$, $t \geq 0$. Namely, due to functional central limit theorem, [2] the corresponding centered and normalized process converges weakly (that is its finite-dimensional distributions converge in Skorokhod topology) to a BM $B_{1/2} = B$, that is

$$\left\{ \frac{N(nt) - \lambda nt}{\sqrt{\lambda n}},\, t \geq 0 \right\} \Rightarrow \{B(t),\, t \geq 0\} \ \text{ as } \ n \to \infty\ . \tag{3.2}$$

This implies the following approximation $N(nt) \approx \lambda nt + \sqrt{\lambda n}B(t)$, which is equivalent to

$$N(t) \approx \lambda t + \sqrt{\lambda}B(t) \ , \tag{3.3}$$

since, by self-similarity, $\sqrt{n}B(t) = B(nt)$. We note that (3.2) holds for general renewal process with mean interrenewal time μ and finite variance σ^2, in which case we replace $\sqrt{\lambda}$ by $\sigma\mu^{-3/2}$ in denominator in (3.2), see [2]. (For Poisson input $\lambda^{-1} = \mu = \sigma$.)

Now we describe an ON/OFF model with m sources and heavy-tailed distributions, which also leads in limit to fractional Brownian traffic [11, 13].

Consider m i.i.d. sources of packet trains, where ON-period is the train length and OFF-period is the intertrain distance. For each source i, the ON-period's lengths $\{\alpha_n^{(i)}, n \geq 1\}$ are i.i.d. and independent of the i.i.d. OFF-period's lengths $\{\tilde{\alpha}_n^{(i)}, n \geq 1\}$. The ON- and OFF-periods are strictly alternating and their lengths have d.f. F_1 and F_2 respectively (for each source). Generally, $F_1 \neq F_2$. We note that for each i the pairs $(\alpha_n^{(i)}, \tilde{\alpha}_n^{(i)})$ form an *alternating renewal process.* Define for each i the binary time series $\{W_i(t), t \geq 0\}$, where $W_i(t) = 1$ if time t belongs to ON-period, and $W_i(t) = 0$ otherwise. Consider the following aggregated cumulative process with a rescaling factor $T > 0$,

$$W_m^*(Tt) = \int_0^{Tt} \sum_{i=1}^{m} W_i(u)du \ , \quad t \geq 0 \ , \tag{3.4}$$

counting packet trains arriving from all m sources in interval $[0, Tt)$. (More exactly, this process counts ON-periods time.)

Let means $\mathbf{E}\,\alpha = \mu_1$, $\mathbf{E}\,\tilde{\alpha} = \mu_2$ be finite, where α, $\tilde{\alpha}$ are corresponding generic variables. The main assumption is that α and $\tilde{\alpha}$ *have infinite second moments* (in other words, exhibit long-range dependence). Namely, we assume that the following *heavy-tailed distribution* assumption

$$1 - F_i(x) \sim c_i x^{-(1+\beta_i)} L_i(x) \ , \quad i = 1, 2 \tag{3.5}$$

is satisfied for F_1 or F_2 (of for both). Here $c_i > 0$ are constants, $0 < \beta_i < 1$ and L_i are slowly varying functions at infinity. It is easy to see that (3.5) does imply finiteness of means but

$$\mathbf{E}\,\alpha^2 = \infty \ , \quad \mathbf{E}\,\tilde{\alpha}^2 = \infty \ .$$

(One can check that $\mathbf{E}\,\alpha^r < \infty$ if $r < 1+\beta_1$, and $\mathbf{E}\,\alpha^r = \infty$ if $r > 1+\beta_1$. Of course, the same holds for variable $\tilde{\alpha}$ with replacement β_1 by β_2, see also [11].) It is assumed that, for each i, the renewal process generated by the ON/OFF-periods is stationary, so in particular,

$$\gamma = \frac{\mu_1}{\mu_1 + \mu_2} \tag{3.6}$$

is the probability to start process with ON-period and, simultaneously it is the time-average limit of the ON-periods time. It is proved in [13] that as $m \to \infty$ and then $T \to \infty$, the following weak convergence

$$\frac{W_m^*(Tt) - Ttm\gamma}{T^H\sqrt{L(T)m}} \Rightarrow cB_H(t) \tag{3.7}$$

holds in the space of *cadlag functions* on the real line (this functional space is equipped with Skorokhod topology). Here $c > 0$ is a constant, B_H is a FBM and L is a slowly varying function (as $T \to \infty$), and c, H, L are completely determined via predefined parameters c_i, β_i, μ_i and functions L_i, $i = 1, 2$.

In work [6], for the same ON/OFF model is found a natural simultaneous time-scaling procedure (there is no need to perform two successive limits in m and T separately). Namely, suppose that d.f. F of (joined) renewal time $\alpha + \tilde{\alpha}$ (that is, F is the convolution of F_1 and F_2) exhibits heavy tail,

$$1 - F(x) \sim x^{-(1+\beta)}L(x)\ , \tag{3.8}$$

where $0 < \beta < 1$, and the number of sources m and rescaling factor T increase in such a way that $m \sim T^\beta$. Then it follows from [6] that the following weak convergence holds:

$$\frac{W_m^*(tm^{1/\beta}) - \gamma tm^{1+1/\beta}}{m^{1/\beta}} \Rightarrow \mathrm{const}\cdot B_H(t)\ , \quad \text{as} \quad m \to \infty\ , \tag{3.9}$$

where $H = 1 - \beta/2$. (This corresponds to (3.7) since $T^H \sim m^{1/\beta - 1/2}$.)

As an application of previous results to Ethernet traffic, we consider a process W_m^* which is the superposition of m i.i.d. renewal processes N_i with Pareto distribution of renewal time with finite mean μ and infinite variance. (That is, the interrenewal time distribution has the form $1 - F(x) = (x_0/x)^\beta$ for $x \geq x_0$ and $0 < \beta < 2$.) That is, process

$$W_m^*(t) = \sum_{i=1}^{m} N_i(t) , \quad t \geq 0$$

counts the accumulated number of (Ethernet) packets generated by m independent identical users in interval $[0, t)$. Then it follows from the above stated results that

$$W_m^*(t) \approx \frac{mt}{\mu} + cm^{1/\beta} B_H(tm^{-1/\beta}) , \tag{3.10}$$

where $c > 0$ is a constant and $H = 1 - \beta/2$. (We note that mt/μ is approximately the mean total number of renewals in $[0, t)$, whereas $tm\gamma$ in (3.7) is (approximately) equal to the mean fraction of ON-periods time in the same interval.) By self-similarity,

$$m^{1/\beta} B_H(tm^{-1/\beta}) = \sqrt{m} B_H(t) \quad \text{in distribution.}$$

This implies the following approximation

$$W_m^*(t) \approx \frac{mt}{\mu} + \sqrt{c^2 m} B_H(t) , \tag{3.11}$$

that corresponds to a fractional Brownian traffic, see (3.1).

In conclusion we compute the variance of an input process $A(t) = \int_0^t \Lambda(u) du$, $t \geq 0$, where Λ is a renewal rate process, [7]. Without loss of generality we assume that $\mathbf{E}\, A(t) = 0$ for all t (see [9]), and let the process Λ be stationary. Applying Fubini's theorem we obtain

$$\begin{aligned} \mathbf{Var}(A(t)) = \mathbf{E}\left(\int_0^t \Lambda(u) du \int_0^t \Lambda(s) ds\right) &= \int_0^t \int_0^t \mathbf{E}(\Lambda(u), \Lambda(s)) duds \\ &= \int_{u=0}^{t} \int_{s=0}^{u} \operatorname{cov}(\Lambda(u), \Lambda(s))\, duds \\ &\quad + \int_{u=0}^{t} \int_{s=u}^{t} \operatorname{cov}(\Lambda(u), \Lambda(s))\, duds . \end{aligned} \tag{3.12}$$

This easily leads to the following result

$$\textbf{Var}(A(t)) = 2\int_{u=0}^{t}\int_{s=0}^{u} \mathrm{cov}\,(\Lambda(u),\Lambda(s))\,duds\;. \tag{3.13}$$

If the renewal rate process is defined as in (2.5) then by stationarity we have (see (2.7))

$$\mathrm{cov}\,(\Lambda(u),\Lambda(s)) = \textbf{Var}(Z_1)\mu^{-1}\int_{|u-s|}^{\infty}(1-F(z))dz\;.$$

We note that if the ON/OFF model has a single source (that is, process Λ is the binary-type process W, see (3.4)) and the following assumptions

$$F_1(t) = t^{-(1+\beta)}L(t)\;,\quad F_2(t) = o(1-F_1(t))\;\text{ as }\; t\to\infty\;,\quad (\beta\in(0,1))$$

hold, then it is shown in [5], that

$$\mathrm{cov}\,(\Lambda(0),\Lambda(t)) \sim \frac{\mu_2^2}{\beta\mu^3}t^{-\beta}L(t)\;,\quad t\to\infty\;,$$

where L is a slowly varying function and $\mu = \mu_1+\mu_2$, see (3.6).

We also note the following useful conservation property of fractional Brownian traffic which is directly deduced from definition 1, [8]: if

$$A_i(t) = m_i t + \sqrt{m_i a}B_H^{(i)}(t)\;,\quad i = 1,\ldots,m$$

are fractional Brownian traffics and FBM's $B_H^{(i)}$ are independent with common parameter H, then the superposition $A(t) = \sum_i A_i(t)$ can be written as $A(t) = mt + \sqrt{ma}B_H(t)$, where $m = \sum_i m_i$ and B_H is a FBM with parameter H.

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